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An Analytical Mathematical Framework to Address the Theta-Logistic Growth Function

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ABSTRACT Most of the growth equations in natural or social science were proposed empirically or based on curved fitting. But there is a lack of unified mathematical foundation. The aim of this study is to derive a basic growth equation so that interrelationship among different growth equations could be understood and the effect of change of one parameter on the other could be easily addressed. Theta-logistic growth function is derived here analytically based on the concept of specific growth rate. The effect of different proposed parameters on growth rate has been represented graphically at the end of the study.

Keywords: Theta-logistic growth function, Von Bertalanffy growth function, Gompertz growth function, Logistic growth function. Specific growth rate

1. Introduction

Growth, an extremely non-linear phenomenon in field of natural science and social science, can be addressed in terms growth functions representing the change in size of a physical quantity with time (or space). Many growth equations [1, 2, 3, 4, 5, 6] have been proposed to describe such growth processes. In the field of biological growth, through only a few have recognized useful because no biologist probably trust that one equation suits all growth processes. 'If physics has its laws, biology has its variety' [7]. The series of existing growth equations brings up a number of questions. Is there any relationship among the equations? Is there any possibility to reduce them to small number of basic forms? How can the shortage or suitability of an equation be interpreted so as to contribute to the understanding of nature?

There are verities of growth functions used to describe growth processes in the field of natural and social science [8,9,10,11,12]. Among them Gompertz growth function [13,14], Von Beratlanffy growth function [15,16,17,18], West type ontogenetic growth function [19,20,21], logistic growth function [22,23], theta-logistic growth function or Richard's growth function [24,25,26,27] have drawn immense attention of the researchers for the years.

Gompertz growth function was introduced to study the mortality rate of human beings [1]. At present, it is extensively used in different fields of natural and social sciences [8,13]. Von Bertalanffy growth function is generally used to describe mostly the growth of plants [2,15]. West-type ontogenetic growth function is helpful to describe the growth of animals [21]. Logistic growth functions are frequently used to study different types of complex nonlinear growth phenomena [11,23]. Discrete logistic growth equations are able to describe chaotic behaviour of the system [28]. The continuous form of logistic growth [6] models does not show intrinsic bifurcations, and as a result, it is much easier to handle analytically. Theta − logistic growth function is also introduced to study different growth processes [3,4,5]. Table 1 shows growth velocity of these growth functions.

Table 1: Some popular growth functions and corresponding growth velocity

Growth function	**Growth velocity**
Gompertz growth function	$\frac{dy}{dt} = y - y\,ln\,y$
Von Beratlanffy growth function	$\frac{dy}{dt} = C_1 y^{0.67} - C_2 y$
West type ontogenetic Growth function	$\frac{dy}{dt} = C_1 y^{0.75} - C_2 y$
logistic growth function	$\frac{dy}{dt} = ry\left(1 - \frac{y}{K}\right)$
Theta-logistic growth function	$\frac{dy}{dt} = ry\left[1 - \left(\frac{y}{K}\right)^{\beta}\right]$

Where, y represents any physical quantity of the system under consideration, K may be termed as carrying capacity, C_1 and C_2 are constants, r is specific growth rate.

Theta-logistic growth function forms the basis of several growth models (e.g.; Von Bertalanffy, Gomperzian, logistic, West-type etc.) [6]. Table – II shows that theta-logistic Growth function forms the basis of different well-known growth functions for different conditions. In this growth function, a new parameter (β) representing intra-specific competition regulates the time required to reach its saturation value. β also indicates nature of fluctuation due to environmental stochasticity. Large fluctuations in population size are observed when β is small, whereas more stable fluctuations are found in case of larger values of β [5]. Table 2 shows the correlation among different growth functions based on the value of β of theta-logistic growth.

Table 2: Theta-logistic growth function and it's correlation with some growth functions

Basic growth function	**Condition**	**Growth function**
Theta-logistic growth function	$\beta = 1$	Logistic
	$\beta \to 0$	Gompertz
	$\beta = -0.34$	Von Bertalanffy
	$\beta = -0.25$	West type biological growth

In general, growth functions are introduced in an empirical manner and do not have theoretical foundation. Some attempts have been made to address these functions from some basic principles [29, 30, 31, 32, 33]. In some cases, at least one assumption is not biologically founded or mathematically correct [34]. Therefore, a biologically convincing, simple and mathematically sound argument for different growth functions is still required. These facts motivated to propose an analytical approach based on change of specific growth rate with time. The proposed framework is able to derive analytically theta-logistic growth function. It is also helpful to consider interrelation among different parameters of theta-logistic growth function.

The paper is organized as follows: In Sec. II, the description of a growth process in a generalized way has been considered and theta-logistic growth function is derived. The correlation among the intrinsic growth rate per capita or specific growth rate, the carrying capacity and the degree of competitive environment has been addressed. In Sec. III, a possible explanation and the effect of variation of the parameters involved in the description of growth processes are given along with graphical presentation. Finally, conclusion is drawn in section IV.

2. The role of specific growth rate and generalized growth equation

The growth dynamics originates from the precise relationship between a physical quantity (x) and it's specific growth rate (r). Therefore, the growth or evolution of any physical quantity can be described with the help of the equation given below:

$$\dot{x} = r(x,t)\,x(t) \qquad (1)$$

where, $r(x,t)$ is the specific growth rate of the variable $x(t)$. $x(t)$ may vary with some other characteristic variable of the system. Here the evolution of a growing system with respect to time is only considered. Thus, an ordinary differential equation, instead of partial, would serve the purpose.

Both, x and r, are function of time. In a growth process, x is expected to increase with time. It is justified to consider that specific growth rate declines with increasing x or with time (t). Thus, it is expected that the rate of change of specific growth rate (i.e, $\dot{r}$) must be less than zero in a growth process $(\dot{r} < 0)$.

Hence, it would be better to introduce a variable $Q(r)$ that can be expressed as

$$Q(r) = -\dot{r} \qquad (2)$$

It is assumed that $Q(r)$ can be expressed in terms of power series as [33],

$$Q(r) = \sum_{i=0}^{\infty} h_i r^i \qquad (3)$$

The polynomial in equation (3) could be truncated up to any terms and it could be used in calculation to represent any growth process. One or more than one terms in that truncated series may be equal to zero. It may be used to derive different growth function analytically.

Here, the truncated series up to third terms has been considered and assumed that the first term (h_0) of the truncated series is zero. Therefore, it generates a quadratic form $Q(r) = h_1 r + h_2 r^2$ as a special case. From equations (1) and (2), the following relation can be established,

$$\ln x = \int r\ dt + \alpha_1 \qquad (4)$$

where, α_1 is a constant of integration.

From equation (1), (2), (3) and (4); the following relation can be established for $Q(r) = h_1 r + h_2 r^2$,

$$x = \left[\left(\frac{x_0^2}{h_1}\right)\left(h_1 + h_2 r_0 - \frac{h_2 r_0}{\exp\,(h_1 t)}\right)\right]^{\frac{1}{h_2}} \qquad (5)$$

and, growth rate ($\dot{x}$) can be expressed as

$$\dot{x} = x_0 r_0 \frac{x^{1-h_2}}{\exp\,(h_1 t)} \qquad (6)$$

where, x_0 and r_0 are constants defined as $x = x_0$ and $r = r_0$ at $t = 0$.

Eliminating t from equation (6) using equation (5), growth rate can be expressed as,

$$\dot{x} = \frac{x}{h_2}\left[x_0^2(h_1 + h_2 r_0)x^{-h_2} - h_1\right] \qquad (7)$$

When $h_2 < 0$, say $h_2 = -\beta$, equation (7) can be expressed as,

$$\dot{x} = \frac{h_1}{\beta} x \left[1 - \frac{x^\beta}{\frac{x_0^\beta h_1}{h_1 - \beta r_0}} \right] \tag{8}$$

When $h_1 - \beta r_0 \neq 0$, equation (8) can be expressed as,

$$\dot{x} = qx \left[1 - \frac{x^\beta}{K^\beta} \right] \tag{9}$$

where, $\frac{x_0^\beta h_1}{h_1 - \beta r_0} = K^\beta$ and $q = \frac{h_1}{\beta}$. q and K may be termed as intrinsic growth rate per capita and carrying capacity respectively. β is related to degree of competitive environment.

It is the form of growth velocity of theta-logistic growth equation. It shows more accurate results than usual logistic growth [4].

2. Result and discussion

The approach proposed here is enriched with several quantitative solutions related to growth mechanisms found often in literature because theta-logistic growth function forms the basis of several growth functions. Here it is possible to consider analytically the effect of change of competitive environment on the carrying capacity and the intrinsic growth rate per capita. It is expected that the state variable (x) must reach a saturation level (x_{max}) for different types of sigmoidal growth for which $x_{max} > 0$ and $r = 0$. To satisfy these conditions, the condition $h_1 > -h_2 r_0$ must be obeyed by the growing system.

Environmental changes and adoptive strategies are well-known to unbalance equillibrium of a system. As a result of such perturbations, a change in h_1 and/or h_2 of the system is expected. The system then would try to reset its saturated value (governed by the value of h_1 and/ or h_2) to a new level and it would show growth (or decay) to attain that level. If the perturbation is within the tolerance limit, the system will follow a new growth dynamic based on the value of h_1 and h_2. Such deviation from saturated level and attainment of new saturation level could be explained by means of proposed unified approach for different growth mechanisms.

Figure 1 shows the variation of growth velocity (dx/dt) with the physical quantity (x) under consideration for $Q(r) = h_1 r + h_2 r^2$ with $x_0 = 0.3$, $r_0 = 73$ and $h_2 = -1.0$ for different values of h_1. It is found that all of the characteristic curves are logistic by nature for $h_2 = -1.0$ .The value of x_{max} decreases with the increase of h_1. A drop in optimum growth velocity is also observed with the increase of h_1.

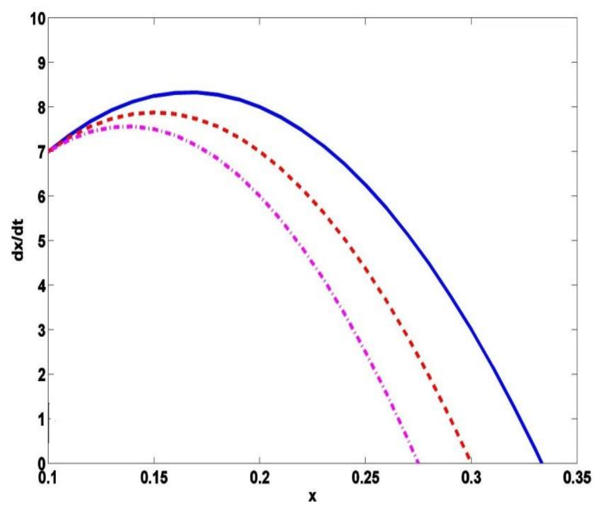

Figure 1: Plots of growth rate (dx/dt) with respect to the state variable x with $x_0 = 0.3$, $r_0 = 73$ and $h_2 = -1.0$ for $Q(r) = h_1 r + h_2 r^2$. From the top, the values of the parameter h_1 are 102, 108 and 112 respectively. All of them represent usual logistic growth by nature. An increase in h_1 lowers the optimum level, the saturation level and the growth rate of state variable x.

Figure 2 shows a comparative representation of Von Bertalanffy type and West type growth for the same h_1 in terms of plots of growth rate (dx/dt) with respect to the state variable x with $x_0 = 0.03$, $r_0 = 15$ for $Q(r) = h_1 r + h_2 r^2$. First (green coloured) and third (black coloured) from the top represent von Bertalanffy type growth ($h_2 = 0.34$) for $h_1 = 0.78$ and $h_1 = 1.12$ respectively. Second (pink coloured) and fourth (blue coloured) characteristics curves from the top represent West type growth ($h_2 = 0.25$) for the same h_1. It is found that the optimum and saturation level of state variable x of von Bertalanffy type growth is less than that of West type growth for same h_1, whereas growth rate of von Bertalanffy type growth is greater than that of West type growth for same h_1.

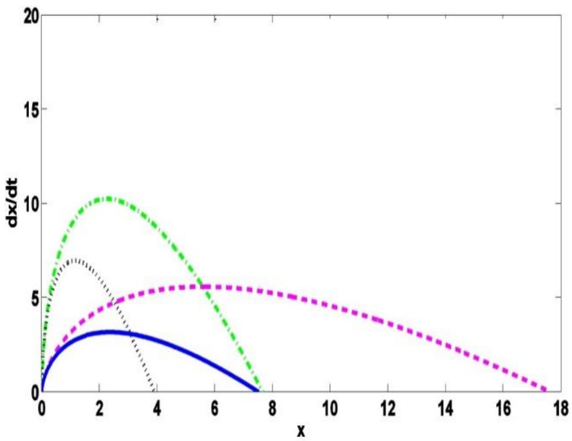

Figure 2: A comparative representation of Von Bertalanffy type and West type growth for the same h_1 in terms of plots of growth rate (dx/dt) with respect to the state variable x with $x_0 = 0.03$, $r_0 = 15$ for $Q(r) = h_1 r + h_2 r^2$. First and third from the top represent Von Bertalanffy type growth ($h_2 = 0.34$) for $h_1 = 0.78$ (green colour) and $h_1 = 1.12$ (black colour) respectively. Second (pink colour) and fourth (blue colour) characteristics curves from the top represent West type growth ($h_2 = 0.25$) for the same h_1. The optimum and saturation level of state variable x of Von Bertalanffy type growth is less than that of West type growth for same h_1, whereas growth rate of Von Bertalanffy type growth is greater than that of West type growth for same h_1.

The differential equation (9) is very similar to theta-logistic growth function. It is not empirically presented in this communication. An analytical approach based on the concept of change of specific growth rate is taken into consideration to derive this expression. The approach can explain the variation of parameters and scaling coefficients of theta-logistic growth function in terms of two coefficients (h_1 and h_2) related to the change of specific growth rate. It correlates maximum attainable value of the state variable (x_{max}) with its specific growth-related coefficients. Therefore, it can be concluded that this analytical approach should be helpful to compare different growth processes in the lime light of theta-logistic growth function. In brief, these are the merits of this communication.

3. Conclusions

The purpose of this communication is to present an analytical framework to derive theta-logistic growth function. It is possible to derive different growth functions from theta-logistic growth function for different limiting condition in the framework described here. The proposed mathematical framework also helps to correlate the intrinsic growth rate per capita, the degree of competitive environment and it carrying capacity. It is also possible to address the change in saturation level of growth processes with the environmental fluctuations. This study may be useful for understanding the basic mechanism behind different biological and social growth processes.

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